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RESEARCH MEMORANDUM

PARALLEL NORMREDUCING TRANSFORMATIONS FOR THE ALGEBRAIC EIGENVALUE PROBLEM

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ABSTRACT

This article presents a unified approach for parallel normreducing methods for the algebraic eigenproblem. The so-called Euclidean parameters presents the problem, to minimize the Frobenius norm of the transform matrix, in a simple form. The use of appropriate preprocessing unitary transforms together with an appropriate pivot strategy leads to convergence to formality.

Keywords: Jacobi methods, parallel transformations, eigenvalues, convergence to normality, Euclidean parameters, normreduction, commutator.

INTRODUCTION

In 1971 Sameh [5] proposed a Jacobi-like eigenvalue algorithm for a parallel computer. Sameh's method is a parallel elaboration of Eberlein's sequentially normreducing transformation procedure [1,4].
In this paper we present a unified approach to Jacobi-like normreducing tranformations and we apply it to elucidate and improve Sameh's method. In a sequentially Jacobi-like procedure for the computation of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of a real or complex matrix $A = A^{(0)}$ (order n) a sequence $\{A^{(j)}\}$ is constructed recursively:

$$A^{(j+1)} = {T^{(j)}}^{-1} A^{(j)} T^{(j)} , \quad j \geqq 0. \tag{1.1}$$

In (1.1) $T^{(j)}$ is a unimodular shear matrix with *Jacobi parameter* $(p_j, q_j, r_j, s_j) \in \mathbb{C}^4$ in (ℓ, m)-*restriction* $\hat{T}_{\ell_j, m_j}$ of $T_{\ell_j, m_j} = T^{(j)}$

$$\hat{T}_{\ell_j, m_j} = \begin{array}{cc} \begin{bmatrix} p_j & q_j \\ r_j & s_j \end{bmatrix} & \begin{array}{l} \longleftarrow \ell_j \\ \longleftarrow m_j \end{array} \\ \begin{array}{cc} \uparrow & \quad\uparrow \\ \ell_j & \quad m_j \end{array} & \end{array} \tag{1.2}$$

In Eberlein-like normreducing processes [1,3,6] the aim is to construct $\{A^{(j)}\}$ so that

$$\lim_{k \to \infty} \| A^{(j)} \|_F = \sum_{j=1}^{n} |\lambda_j|^2 , \tag{1.3}$$

that means $\{A^{(j)}\}$ *converges to normality* [2]. Eberlein [1] gives for each iteration an approximation of the optimal normreducing $T_{\ell,m}$. These choices of the Jacobi parameters, together with a well-defined pivot strategy $\{(\ell_j, m_j)\}$ brings about convergence to normality.
Since the Frobenius norm is invariant under unitary transformations, the optimal norm-reducing shear $T_{\ell,m}$ is determined except for a unitary factor. Hence we discuss the normreduction in theoretical terms that are invariant under unitary transformations.

Matrices $S,P \in \mathbb{C}^{n\times n}$ will be called *row-congruent* if and only if $S = PU$ for some unitary U, notation $S \sim P$. It is easy to see that $S \sim P$ if and only if $SS^* = PP^*$. Now for shear $T_{\ell,m}$, with $\hat{T}_{\ell,m} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

$$\hat{T}_{\ell,m}\hat{T}^*_{\ell,m} \begin{bmatrix} |p|^2 + |q|^2 & p\bar{r} + q\bar{s} \\ \bar{p}r + \bar{q}s & |r|^2 + |s|^2 \end{bmatrix} = \begin{bmatrix} x & z \\ \bar{z} & y \end{bmatrix} \tag{1.3}$$

The quantities

$$x = x(T_{\ell,m}) = |p|^2 + |q|^2,\ y = y(T_{\ell,m)} = |r|^2 + |s|^2\ ,$$

$$z = z(T_{\ell,m}) = u + iv = p\bar{r} + q\bar{s} \tag{1.4}$$

will be called the *Euclidean parameters* of $T_{\ell,m}$ [3]. These parameters are pre-eminent appropriate for the formulation of the Frobenius norm of $T^{-1}AT$.

We assume $T_{\ell,m}$ to be unimodular, so

$$x,y > 0 \text{ and } |ps - qr|^2 = xy - |z|^2 = 1 \tag{1.4}$$

With $z = u + iv$

$$\mathcal{H} = \{(x,y,u,v) \in \mathbb{R}^4 | xy-u^2-v^2 = 1,\ x,y > 0\} \tag{1.5}$$

is the *positive sheet* of the elliptic hyperboloid $xy-u^2-v^2 = 1$. In case of real norm-reducing shears, $\hat{T}_{\ell,m} \in \mathbb{R}^{2\times 2}$,

$$x = p^2 + q^2,\ y = r^2 + s^2,\ z = pr + qs,\ xy - z^2 = 1. \tag{1.6}$$

The Euclidean paramters x, y and z of that unimodular shear $T_{\ell,m}$ correspond with the points in the positive sheet of $xy - z^2 = 1$ in $\mathbb{R}^3$.

In the *parallel normreduction* $A^{(j)}$ is transformed by a direct sum of *identical unimodular* 2×2 *matrices*:

$$A^{(j+1)} = W_j^{-1} A^{(j)} W_j \ , \quad j \geqq 0 \ , \tag{1.7}$$

where $W_j = \mathrm{diag}(T_{1,j}, \ldots, T_{k,j})$, $n = 2k$ and

$$T = T_{i,j} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \ , \quad i = 1, \ldots, k. \tag{1.8}$$

Then $\|A^{(j+1)}\|_F^2$ is a quadratic function of the Euclidean parameters x, y, u and v ($z = v + iv$) of the k matrices $T_{i,j}$ in W_j. The minimization of that quartic on $\mathcal{H}$ leads to a generalized eigenvalue problem in four dimensions. Section 2 describes the first step $A^{(1)} = W^{-1}AW$ of the normreducing process for real matrices, and there is shown the *commutator* $(c_{i,j}^{(1)}) = C^{(1)} = A^{(1)^T}A^{(1)} - A^{(1)}A^{(1)^T}$ in relation to parallel shear transformations: $c_{2\ell-1,2\ell}^{(1)} = 0$ and $c_{2\ell-1,2\ell-1}^{(1)} = c_{2\ell,2\ell}^{(1)}$, $\ell = 1, \ldots, k$ iff transformation W minimizes $\|A^{(1)}\|_F$. This section gives also the construction of the optimal W. A special analysis is given to the step in which $\inf\{\|W_j^{-1}A^{(j)}W_j\|_F \mid W = T_{1,j} \oplus \ldots \oplus T_{k,j}\}$ is not assumed for unimodular 2×2 shear $T_{i,j}$. Section 3 describes the same problems for complex matrices. In section 4 will be shown that a well-chosen pivot strategy $\{(\ell_{i,j}, m_{i,j}) \mid i = 1, \ldots, k\}_{j=0}^{\infty}$ together with an appropriate preprocessing sequence of unitary matrices $U^{(j)}$ results in sequence $\{A^{(j)}\}$ that converges to normality.

2. PARALLEL NORMREDUCTION: REAL MATRICES

Let the matrix S be real and of even order $n = 2k$. Then it can be partitioned as follows

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & & & \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix} \tag{2.1}$$

where each submatrix is given by

$$A_{\ell,m} = \begin{bmatrix} a_{2\ell-1,2m-1} & a_{2\ell-1,2m} \\ a_{2\ell,2m-1} & a_{2\ell,2m} \end{bmatrix}, \quad \ell,m = 1,\dots,k. \tag{2.2}$$

For convenience define

$$\begin{aligned} &\sigma_{\ell,m} = a_{2\ell,2m-1},\ \mu_{\ell,m} = a_{2\ell-1,2m},\ \alpha_{\ell,m} = a_{2\ell-1,2m-1}, \\ &\beta_{\ell,m} = a_{2\ell,2m},\ \nu_{\ell,m} = \alpha_{\ell,m} - \beta_{\ell,m}, \quad \ell,m = 1,\dots,k. \end{aligned} \tag{2.3}$$

Let

$$A' = W^{-1}AW , \tag{2.4}$$

where

$$W = \mathrm{diag}(S_1,S_2,\dots,S_k)$$

with

$$S_j = \begin{bmatrix} p_j & q_j \\ r_j & s_j \end{bmatrix}, \ p_js_j - q_jr_j = 1 , \quad j = 1,\dots,k.$$

We introduce

$$x_j = p_j^2 + q_j^2,\ y_j = x_j^2 + s_j^2,\ z_j = p_j r_j + q_j s_j\ , \qquad j = 1,\dots,k;$$

they are the Euclidean parameters of the unimodular S_j. So

$$x_j y_j - z_j^2 = 1\ , \qquad i = 1,\dots,k.$$

Then

$$A'_{\ell,m} = S_\ell^{-1} A_{\ell,m} S_m\ .$$

Easy calculations give the following results.

THEOREM 2.1. For each (ℓ,m), $\ell,m = 1,\dots,k$, $\|A'_{\ell,m}\|_F^2$ is a bilinear function of the Euclidean parameters (x_ℓ,y_ℓ,z_ℓ) of S_ℓ and (x_m,y_m,z_m) of S_m:

$$\|A'_{\ell,m}\|_F^2 = (x_\ell,y_\ell,z_\ell)\begin{bmatrix} \sigma_{\ell,m}^2 & \beta_{\ell,m}^2 & 2\sigma_{\ell,m}\beta_{\ell,m} \\ \alpha_{\ell,m}^2 & \mu_{\ell,m}^2 & 2\alpha_{\ell,m}\mu_{\ell,m} \\ -2\alpha_{\ell,m}\sigma_{\ell,m} & -2\beta_{\ell,m}\mu_{\ell,m} & -2(\alpha_{\ell,m}\beta_{\ell,m}+\sigma_{\ell,m}\mu_{\ell,m}) \end{bmatrix}\begin{bmatrix} x_m \\ y_m \\ z_m \end{bmatrix}. \ \square$$

THEOREM 2.2. For each $j = 1,\dots,k$ let be

$$w_j = (x_j-y_j)/2,\ t_j = (x_j+y_j)/2 = (1+w_j^2+z_j^2)^{\frac{1}{2}}. \tag{2.5}$$

Then $\|W^{-1}AW\|_F^2$ is a quadratic function $(\mathbf{w},\mathbf{z}) \longmapsto g(\mathbf{w},\mathbf{z};A)$ where $\mathbf{w} = (w_1,\dots,w_k)^T$ and $\mathbf{z} = (z_1,\dots,z_k)^T$. Moreover

$$\frac{\partial g}{\partial w_\ell}(0,0;A) = c_{2\ell-1,2\ell+1} - c_{2\ell,2\ell},\ \frac{\partial g}{\partial z_\ell}(0,0;A) = c_{2\ell-1,2\ell}\ ,$$

where $(c_{i,j}) = A^TA - AA^T$. $\square$

The complexity of the unconstrained minimization of g forces the restriction to a problem with fewer degrees of freedom. Therefore, we consider, as in [5]

$$W = \mathrm{diag}(S_1,\dots,S_k) \tag{2.6}$$

with

$$S_j = S = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad ps - qr = 1, \quad j = 1,\dots,k. \tag{2.7}$$

Such a matrix W will be called a *diagonal of shears*. Now

$$x = p^2 + q^2, \; y = r^2 + s^2, \; z = pr - qs. \tag{2.8}$$

The unimodularity of S implies

$$xy - z^2 = 1. \tag{2.9}$$

THEOREM 2.3. If W is a diagonal of shears with Euclidean parameters (x,y,z) then

$$\|W^{-1}AW\|_F^2 = \kappa + \sum_{\ell,m=1}^{k} (-\sigma_{\ell,m}x + \mu_{\ell,m}y + \nu_{\ell,m}z)^2, \tag{2.10}$$

where $\kappa = \sum_{\ell,m=1}^{k} (\mathrm{tr}(A_{\ell,m})^2 - 2\det(A_{\ell,m}))$. □

As in (2.5) we define

$$w := (x-y)/2, \; t := (x+y)/2 = (1+w^2+z^2)^{\frac{1}{2}}.$$

Then, as follows from (2.10), $\|W^{-1}AW\|_F^2$ is a function of w and z:

$$g(w,z;A) := \|W^{-1}AW\|_F^2. \tag{2.11}$$

With simple but cumbersome calculations one proves the following lemma.

LEMMA 2.4. Let be $(c'_{i,j}) = (A')^{\mathsf T}A' - A'(A')^{\mathsf T}$, where $A' = W^{-1}AW$ with W as defined in (2.6) and (2.7). Then

$$\sum_{\ell=1}^{k}\begin{bmatrix} c'_{2\ell-1,2\ell-1} - c'_{2\ell,2\ell} \\ \\ 2c'_{2\ell-1,2\ell-1} \end{bmatrix} = \begin{bmatrix} (p^2+s^2-q^2-r^2)/2 & pr-qs \\ \\ pq - rs & ps+qr \end{bmatrix}\begin{bmatrix} \frac{\partial g}{\partial w}(w,z;A) \\ \\ \frac{\partial g}{\partial z}(w,z;A) \end{bmatrix}, \tag{2.12}$$

where g as defined in (2.11). □

THEOREM 2.5. The function g is stationary in $(w,z) \in \mathbb{R}^2$ iff

$$\sum_{\ell=1}^{k}(c'_{2\ell-1,2\ell-1} - c'_{2\ell,2\ell}) = 0 \text{ and } \sum_{\ell=1}^{k}(c'_{2\ell-1,2\ell}) = 0.$$

PROOF. The determinant of the coefficientmatrix in (2.11) equals $\frac{1}{2}(x+y)(ps-qr) \neq 0$. □

Theorem 2.3 implies that the determination of the optimale normreducing diagonal of shears requires the minimization of a quadratic function subject to $xy - z^2 = 1$. Let be

$$\mathbf{d} = (d_1,d_2,d_3)^T = (x,y,z),$$

$$H := \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{2.13}$$

and

$$\mathcal{H} := \{\mathbf{d} \in \mathbb{R}^3 | \mathbf{d}^T H \mathbf{d} = 1,\ d_1 > 0 \}. \tag{2.14}$$

Further

$$B = (\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3) \in \mathbb{R}^{k^2 \times 3}$$

with

$$\mathbf{b}_1 = (\sigma_{1,1},\sigma_{1,2},\ldots,\sigma_{k,k})^\mathsf{T} \in \mathbb{R}^{k^2}$$

$$\mathbf{b}_2 = (\mu_{1,1},\mu_{1,2},\ldots,\mu_{k,k})^\mathsf{T} \in \mathbb{R}^{k^2}$$

$$\mathbf{b}_3 = (\nu_{1,1},\nu_{1,2},\ldots,\nu_{k,k})^\mathsf{T} \in \mathbb{R}^{k^2}$$

The Euclidean parameters $\mathbf{d} = (x,y,z)$ of an optimal normreducing diagonal of shears solve the problem

$$\min\{\|B\mathbf{d}\| \,|\, \mathbf{d}^\mathsf{T}H\mathbf{d} = 1\}. \tag{2.15}$$

(2.15) leads to a gerenalized eigenproblem in three dimensions. The analysis for the three cases rank(B) equals 3, 2 or 1 is summarized in the following theorems.

THEOREM 2.6. Let be rank(B) = 3, and $B = QR$ with $Q \in \mathbb{R}^{k^2\times 3}$ orthogonal and $R \in \mathbb{R}^{3\times 3}$ uppertriangular. Then $\|B\mathbf{d}\|$ assumes its minimum on $\mathcal{H}$ in $\tilde{\mathbf{d}}$, where $R\tilde{\mathbf{d}}$ is an eigenvector corresponding with the unique positive eigenvector of $R^{-1}HR$.

PROOF. The existence of a minimum follows from compactness and continuity arguments. $(B^\mathsf{T}B-\lambda H)\mathbf{d} = \mathbf{0}$ implies $(R^{-\mathsf{T}}HR^{-1}-\rho^{-1}I)R\mathbf{d} = \mathbf{0}$. The eigenvalue ρ^{-1} corresponding with the minimum is positive on base of convexity arguments. The positive eigenvalue of $R^{-\mathsf{T}}HR^{-1}$ is unique. □

THEOREM 2.7. Let be rank(B) = 1 and range$(B^\mathsf{T}) = \text{span}(-\sigma,\mu,\nu)^\mathsf{T}$. Then

(i) $\min\{\|B\mathbf{d}\| \,|\, \mathbf{d} \in \mathcal{H}\} = 0$, $\nu^2 + 4\sigma\mu > 0$ or $\nu = \sigma = \mu = 0$;

(ii) $\min\{\|B\mathbf{d}\| \,|\, \mathbf{d} \in \mathcal{H}\} = |\nu^2+4\sigma\mu|$, $\nu^2 + 4\sigma\mu < 0$;

(iii) $\inf\{\|B\mathbf{d}\| \,|\, \mathbf{d} \in \mathcal{H}\} = 0$, $\nu^2 + 4\sigma\mu = 0 \wedge |\sigma| + |\mu| \neq 0$,

in this case the infimum is not assumed.

PROOF. In A each block $A_{\ell,m}$ gives a similar contribution to $B\mathbf{d}$; hence $B\mathbf{d}$ can be considered to come from one block, say $C = \begin{bmatrix} \alpha & \mu \\ \sigma & \beta \end{bmatrix}$.

(i) C is diagonalizable with a real shear. The equation $-\sigma d_1 + \mu d_2 + \nu d_3 = 0$ determines a solution curve Γ in $\mathcal{H}$. The parametric form of Γ is

$$\mathbf{p}(\tau) = (\tau, \tau^{-1}, 0)^T \quad , \sigma = \mu = 0 \quad (p_1 > 0)$$

$$\mathbf{p}(\tau) = (\nu\tau/\sigma,\ \sigma(1+\tau^2)(\nu\tau)^{-1},\ \tau)^T \quad , \sigma\nu \neq 0,\ \mu = 0 \quad (p_1 > 0)$$

$$\mathbf{p}(\tau) = (-\mu(1+\tau^2)(\nu\tau)^{-1}\ , -\nu\tau/\mu,\ \tau)^T \ , \mu\nu \neq 0,\ \lambda = 0 \quad (p_1 > 0)$$

$$\mathbf{p}(\tau) = (\frac{\nu\tau \pm D}{2\sigma}, \frac{-\nu\tau \mp D}{2\mu}, \tau)^T \quad , D^2 = (\nu^2 + 4\sigma\mu)\tau^2 + 4\sigma\mu > 0,\ \lambda\mu \neq 0.$$

(ii) Transform C in a Murnaghan form. The positive minimum is assumed for

$$\mathbf{d} = -|4\sigma\mu + \nu^2|^{\frac{1}{2}}\sigma/|\sigma|(-2\sigma, 2\mu, -\nu)^T.$$

(iii) C is not diagonalizable. B^TB has no eigenvector in $\mathcal{H}$. The plane $-\sigma d_1 + \mu d_2 + \nu d_3 = 0$ contacts the recession cone $\{\mathbf{d} | d_1 d_2 - d_3^2 = 0, d_1 > 0\}$ of $\mathcal{H}$ along the line $\mathcal{L} : \mathbf{p}(\tau) = \tau(2\mu, -2\sigma, -\nu)^T$. Hence $\|B\mathbf{d}\| > 0$ on $\mathcal{H}$, for $\mathcal{L} \cap \mathcal{H} = \emptyset$. Now we describe a curve Γ on $\mathcal{H}$ such that $\mathcal{L}$ is its asymptote:

$$\Gamma : \mathbf{d}(\tau) = \tfrac{1}{2}\tau(\mu-\sigma)^{-1}(2\mu, -2\sigma, -\nu) + \tfrac{1}{2}\tau^{-1}(\mu-\sigma)^{-1}(-2\sigma, 2\mu, \nu),\ \tau > 0. \tag{2.16}$$

On Γ we find, using the fact that $\nu^2 + 4\sigma\mu = 0$

$$-\sigma d_1(\tau) + \mu d_2(\tau) + \nu d_3(\tau) = (\mu-\sigma)/\tau \to 0,\ \tau \to \infty.$$

The infimum zero of $\|B\mathbf{d}\|$ on $\mathcal{H}$ is not assumed. □

EXAMPLE 2.1.

$$A = \begin{bmatrix} -1 & 1 & -2 & 2 \\ -1 & 1 & -2 & 2 \\ -3 & 3 & -4 & 4 \\ -3 & 3 & -4 & 4 \end{bmatrix}$$

Then

$$B^{\top} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \end{bmatrix} \text{ and } H^{-1}B^{\top}B = 60\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix}$$

Remark that $\mathrm{tr}(H^{-1}B^{\top}B) = -(4\sigma\mu+\nu^2) = 0$.
The threefold eigenvalue 0 of nondiagonalizable $H^{-1}B^{\top}B$ gives eigenvectors $(-\gamma_1+2\gamma_2,\gamma_1,\gamma_2)^{\top}$, none in $\mathcal{H}$, for $(-\gamma_1+2\gamma_2)\gamma_1 - \gamma_2^2 = -(\gamma_1-\gamma_2)^2 \leqq 0$. With (2.16) we get, since $\sigma = -1$, $\mu = 1$, $\nu = -2$.

$$\Gamma : \mathbf{d}(\tau) = \tfrac{1}{2}(\tau+\tau^{-1},\ \tau+\tau^{-1},\ t-\tau^{-1}) \in \mathcal{H},\ \tau > 0.$$

Along Γ holds $\|B\mathbf{d}(\tau)\| \to 0$ $(,\tau \to 0)$. □

In a similar way one derives the next theorem for the case rank(B) = 2.

THEOREM 2.8. Let be rank(B) = 2 and nullity(B) = $\mathrm{span}(t_1,t_2,t_3)^{\top}$. Then

(i) $\min\{\|B\mathbf{d}\| \,|\, \mathbf{d} \in \mathcal{H}\} = 0$, $t_1t_2 > t_3^2$;

(ii) $\min\{\|B\mathbf{d}\| \,|\, \mathbf{d} \in \mathcal{H}\} > 0$, $t_1t_2 < t_3^2$;

(iii) $\inf\{\|B\mathbf{d}\| \,|\, \mathbf{d} \in \mathcal{H}\} = 0$, $t_1t_2 = t_3^2$. This infimum is not assumed for finite $\mathbf{d} \in \mathcal{H}$. □

REMARK. In case (iii) $H^{-1}B^{\top}B$ is not diagonalizable. The algebraic multiplicity of eigenvalue zero of $H^{-1}B^{\top}B$ equals two as can be seen from the coefficient of the first grade term in the characteristic polynomial of $H^{-1}B^{T}B$, being

$$(t_3^2-t_1t_2)\left[\left[\sum_{i=1}^{k}\mu_i\nu_i\right]^2 - \sum_{i=1}^{k}\mu_i \sum_{i=1}^{k}\nu_i\right]/t_1^2 = 0. \ \square$$

EXAMPLE 2.2.

$$A = \begin{bmatrix} 2 & -2 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ 4 & -4 & 9 & -8 \\ -3 & 0 & -6 & 1 \end{bmatrix}$$

Then

$$B^{T} = \begin{bmatrix} 1 & 2 & 3 & 6 \\ -2 & -4 & -4 & -8 \\ 1 & 2 & 4 & 8 \end{bmatrix}; \ \mathrm{rank}(B) = 2, \ H^{-1}B^{T}B = \begin{bmatrix} -140 & 200 & -180 \\ 100 & -140 & 130 \\ -65 & 90 & -85 \end{bmatrix}.$$

$\mathcal{N}(B) = \mathrm{span}(4,1,-2)^{T}$. Along curve Γ in $\mathcal{H}$,

$$\Gamma : \mathbf{d}(\tau) = \tfrac{1}{2}\tau(3 + 5\sqrt{1+4\tau^{-2}/25}, \ -3+5\sqrt{1+4\tau^{-2}/25}, \ -4\tau)^{T}, \ \tau > 0$$

holds: $\lim_{\tau\to\infty} \tau^{-1}\mathbf{d}(\tau) = (4,1,-2)^{T}$ and

$$\|B\mathbf{d}(\tau)\|^2 = \frac{32}{5}\,(1 + \sqrt{1+4\tau^{-2}/25})^{-2}\,\tau^{-4} \to 0 \ (\tau \to \infty).$$

$\mathcal{N}(B)$ is the eigenspace of the eigenvalue zero of $H\ \ B^{T}B$. $\square$

3. PARALLEL NORMREDUCTION: COMPLEX MATRICES

Let be A a complex matrix of even order $n = 2k$. The partitioning of A and the notations are as in (2.1) until (2.4). Now we denote

$$x_j = |p_j|^2+|q_j|^2,\ y_j = |r_j|^2+|s_j|^2,\ z_j = p_j\bar{r}_j+q_j\bar{s}_j,\ j = 1,\ldots,k\ , \tag{3.1}$$

where p_j,q_j,r_j,s_j are the Jacobi parameters of the complex shear S_j in W. As a consequence of the unimodularity of S_j:

$$x_jy_j - |z_j|^2 = 1\ , \quad j = 1,\ldots,k\ . \tag{3.2}$$

By simple calculations one derives the following theorem.

THEOREM 3.1. Let be $A' = W^{-1}AW$, $W = \mathrm{diag}(S_1,\ldots,S_k)$. Then $\|A'\|_F^2$ is a quadratic function of x_j, y_j, z_j and $\bar{z}_j$, namely

$$\|A'\|_F^2 = \sum_{\ell,m=1}^{k} (x_\ell,y_\ell,z_\ell,\bar{z}_\ell)B_{\ell,m}(x_m,y_m,z_m,\bar{z}_m)^T =$$

where

$$B_{\ell,m} = \begin{bmatrix} |\sigma_{\ell,m}|^2 & |\beta_{\ell,m}|^2 & \sigma_{\ell,m}\bar{\beta}_{\ell,m} & \bar{\sigma}_{\ell,m}\beta_{\ell,m} \\ |\alpha_{\ell,m}|^2 & |\mu_{\ell,m}|^2 & \alpha_{\ell,m}\bar{\mu}_{\ell,m} & \bar{\alpha}_{\ell,m}\mu_{\ell,m} \\ -\bar{\alpha}_{\ell,m}\sigma_{\ell,m} & -\bar{\mu}_{\ell,m}\beta_{\ell,m} & -\sigma_{\ell,m}\bar{\mu}_{\ell,m} & -\bar{\alpha}_{\ell,m}\beta_{\ell,m} \\ -\alpha_{\ell,m}\bar{\sigma}_{\ell,m} & -\mu_{\ell,m}\bar{\beta}_{\ell,m} & -\alpha_{\ell,m}\beta_{\ell,m} & -\bar{\sigma}_{\ell,m}\mu_{\ell,m} \end{bmatrix}. \tag{3.3}$$

□

Let be

$$x_j = t_j + w_j,\quad y_j = t_j - w_j,\ j = 1,\ldots,k \tag{3.4}$$

Then, as follows from the unimodularity of S_j,

$$t_j = (1 + w_j^2 + z_j\bar{z}_j)^{\frac{1}{2}}.$$

Hence $\|A'\|_F^2$ is a function of the k triples $(w_j,z_j,\bar{z}_j)$, $j = 1,\dots,k$:

$$\|A'\|_F^2 = g(w_1,z_1,\bar{z}_1,\dots,w_k,z_k,\bar{z}_k).$$

The relation between the commutar $C^{(')} = A^{(')*}A^{(')}-A^{(')}A^{(')*}$ and g is mentioned in

THEOREM 3.2. The partial derivatives of g in $\mathbf{0} = (0,0,0,\dots,0,0,0) \in \mathbb{C}^{3k}$ satisfy the following densities

$$\begin{cases} \dfrac{\partial g}{\partial w_j}(\mathbf{0}) = c_{2j-1,2j-1} - c_{2j,2j}, \\ \dfrac{\partial g}{\partial z_j}(\mathbf{0}) = c_{2j,2j-1}, \qquad\qquad j = 1,\dots,k \\ \dfrac{\partial g}{\partial \bar{z}_j}(\mathbf{0}) = c_{2j-1,2j}. \end{cases} \tag{3.5}$$

PROOF. Use the properties

$$\frac{\partial x_j}{\partial w_j} = 1 + w_j/t_j = x_j/t_j,\ \frac{\partial y_j}{\partial w_j} = 1 - w_j/t_j = y_j/t_j,\ \frac{\partial x_j}{\partial z_j} = \bar{z}_j/t_j$$

and

$$c_{2j-1,2j-1} - c_{2j,2j} = \sum_{m=1}^{k}(|\alpha_{m,j}|^2 + |\lambda_{m,j}|^2 - |\mu_{m,j}|^2 - |\beta_{m,j}|^2 +$$
$$|\lambda_{j,m}|^2 + |\beta_{j,m}|^2 - |\alpha_{j,m}|^2 - |\mu_{j,m}|^2),$$

$$c_{2j-1,2j} = \sum_{m=1}^{k}(\bar{\alpha}_{m,j}\mu_{m,j} + \bar{\lambda}_{m,j}\beta_{m,j} - \alpha_{j,m}\bar{\lambda}_{j,m} - \mu_{j,m}\bar{\beta}_{j,m}).$$

With (3.3), and (3.4) and easy but cumbersome calculation one finds (3.5). □

As in the preceding section we restrict ourselves to a *diagonal of shears*:

$$W = \mathrm{diag}(S_1,\dots,S_k) \tag{3.6}$$

with

$$S_j = S = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \; ps - qr = 1, \; j = 1,\dots,k. \tag{3.7}$$

Similar to (2.8) we define common Euclidean parameters

$$x = |p|^2 + |q|^2, \; y = |r|^2 + |s|^2, \; z = p\bar{r} + q\bar{s} = u + iv \tag{3.8}$$

with

$$xy - |z|^2 = 1 , \tag{3.9}$$

where $|z|^2 = u^2 + v^2$. Further

$$\mathcal{H} = \{(x,y,u,v)\,|\,x > 0, \; xy - u^2 - v^2 = 1\} .$$

By rather simple calculations $\|W^{-1}AW\|_F^2$ appears to be a quadratic function of x, y, u and v.

THEOREM 3.3. If W is a diagonal of shears with common Euclidean parameters $x,y,z = u + iv$ then $\|W^{-1}AW\|_F^2$ is expressible in terms of these parameters, viz.

$$\|W^{-1}AW\|_F^2 = f(x,y,z) = \sum_{\ell,m=1}^{k} (|\alpha_{\ell,m}|^2 + |\beta_{\ell,m}|^2 + |\mu_{\ell,m}y + \nu_{\ell,m}z|^2$$

$$+ |\sigma_{\ell,m}x - \nu_{\ell,m}\bar{z}|^2 - |\nu_{\ell,m}z|^2 - 2\,\mathrm{Re}(\bar{\sigma}_{\ell,m}\mu_{\ell,m}z^2) =$$

$$= \sum_{\ell,m=1}^{k} (|\alpha_{\ell,m}|^2 + |\beta_{\ell,m}|^2) + (x,y,u,v) \sum_{\ell,m=1}^{k} P_{\ell,m}(x,y,u,v)^{\mathsf{T}}, \tag{3.10}$$

where

$$\begin{bmatrix} |\sigma_{\ell,m}|^2 & 0 & -\mathrm{Re}(\bar{\sigma}_{\ell,m}\nu_{\ell,m}) & -\mathrm{Im}(\bar{\sigma}_{\ell,m}\nu_{\ell,m}) \\ 0 & |\mu_{\ell,m}|^2 & \mathrm{Re}(\mu_{\ell,m}\bar{\nu}_{\ell,m}) & \mathrm{Im}(\mu_{\ell,m}\bar{\nu}_{\ell,m}) \\ -\mathrm{Re}(\bar{\sigma}_{\ell,m}\nu_{\ell,m}) & \mathrm{Re}(\mu_{\ell,m}\bar{\nu}_{\ell,m}) & |\nu_{\ell,m}|^2-2\mathrm{Re}(\sigma_{\ell,m}\bar{\mu}_{\ell,m}) & 2\mathrm{Im}(\sigma_{\ell,m}\bar{\mu}_{\ell,m}) \\ -\mathrm{Im}(\bar{\sigma}_{\ell,m}\nu_{\ell,m}) & \mathrm{Im}(\mu_{\ell,m}\bar{\nu}_{\ell,m}) & 2\mathrm{Im}(\sigma_{\ell,m},\bar{\mu}_{\ell,m}) & |\nu_{\ell,m}|^2+2\mathrm{Re}(\sigma_{\ell,m}\bar{\mu}_{\ell,m}) \end{bmatrix} \quad \square \qquad (3.11)$$

Analogously to section 2 we define

$$w = (x-g)/2, \; t = (x+y)/2 = \sqrt{1+u^2+v^2+w^2} \; . \qquad (3.12)$$

Then, as follows from (3.10), $\|W^{-1}AW\|_F^2$ is a function of w, u and v:

$$g(w,u,v;A) := \|W^{-1}AW\|_F^2 \; . \qquad (3.13)$$

With simple but cumbersome calculations one proves the following lemma.

LEMMA 3.4. Let be $(c'_{i,j}) = (A')^*A' - A'(A')^*$, where $A' = W^{-1}AW$ with W as defined in (3.6) and (3.7). Then

$$\sum_{\ell=1}^{k} \begin{bmatrix} c'_{2\ell-1,2\ell-1} - c'_{2\ell,2\ell} \\ c'_{2\ell-1,2\ell} \\ c'_{2\ell,2\ell-1} \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} |p|^2+|s|^2-|q|^2-|r|^2 & \bar{p}r-\bar{q}s & p\bar{r}-q\bar{s} \\ \bar{p}q - \bar{r}s & \bar{p}s & q\bar{r} \\ p\bar{q} - r\bar{s} & \bar{q}r & p\bar{s} \end{bmatrix} \begin{bmatrix} g_w \\ g_u+ig_v \\ g_u-ig_v \end{bmatrix} \qquad (3.14)$$

where g as defined in (3.11). □

THEOREM 3.5. The function g is stationary in $(w,u,v) \in \mathbb{R}^3$ iff

$$\sum_{\ell=1}^{k} (c'_{2\ell-1,2\ell-1} - c'_{2\ell,2\ell}) = 0 \text{ and } \sum_{\ell=1}^{k} c'_{2\ell-1,2\ell} = 0.$$

PROOF. The determinant of the coefficient matrix in (3.12) equals $\tfrac{1}{2}|ps - qr|^2(|p|^2+|q|^2+|r|^2+|s|^2) = t > 0$. This proofs the theorem. □

Theorem 3.3 implies that the determination of the optimal normreducing diagonal of shears requires the minimization of a quadratic function subject to $xy - u^2 - v^2 = 1$. Neither the function f in (3.10) nor $h(x,y,z) = xy - |z|^2$ are analytic, in contrast with the corresponding real functions (2.10) and (2.9) resp.

Let be $\mathbf{d} = (d_1, d_2, d_3, {}_4) = (x,y,u,v)$,

$$H = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.15}$$

and

$$\mathcal{H} = \{\mathbf{d} \in \mathbf{R}^4 \,|\, \mathbf{d}^\top H\mathbf{d} = 1,\ d_1 > 0\} \tag{3.16}$$

Further $M = \sum_{\ell,m=1}^{k} P_{\ell,m}$, where $P_{\ell,m} \in \mathbf{R}^{4\times 4}$ defined in theorem 3.3 and let be $h(\mathbf{d}) := \mathbf{d}^\top M\mathbf{d},\ \mathbf{d} \in \mathbf{R}^4$.

With these notations the minimization problem becomes: solve

$$\min\{h(\mathbf{d}) \,|\, \mathbf{d} \in \mathcal{H}\}\ . \tag{3.17}$$

Let be λ_j, $j = 1,\dots,n$ the eigenvalues of A. Then

$$\sum_{j=1}^{n} |\lambda_j|^2 \leqq \|W^{-1}AW\|_F^2 = h(\mathbf{d}) + \sum_{\ell,m=1}^{k} (|\alpha_{\ell,m}|^2 + |\beta_{\ell,m}|^2)$$

for each $\mathbf{d} \in \mathcal{H}$. So h is bounded below on $\mathcal{H}$. Let be

$$\eta = \inf\{h(\mathbf{d}) \,|\, \mathbf{d} \in \mathcal{H}\}\ . \tag{3.18}$$

THEOREM 3.6. Let be η as defined in (3.18).

(i) If infimum η is assumed in $\mathbf{d} \in \mathcal{H}$ then $M\mathbf{d} = \eta H\mathbf{d}$.

(ii) If $h(\mathbf{x}) > \eta$ for each $\mathbf{x} \in \mathcal{H}$ then there exist a $\mathbf{d} = (d_1, d_2, d_3, d_4)^\top$ such that $M\mathbf{d} = \eta H\mathbf{d}$, $\|\mathbf{d}\| = 1$, $d_1 > 0$ and $\mathbf{d}^\top H\mathbf{d} = 0$.

PROOF.

(i) The Lagrange multiplier method gives $\mathbf{Md} = \lambda\mathbf{Hd}$ and $\eta = \mathbf{d}^T\mathbf{Md} = \lambda\mathbf{d}^T\mathrm{H}\mathbf{d} = \lambda$. So $\mathbf{Md} = \eta\mathbf{Hd}$.

(ii) Let be $\mathcal{K} = \{\mathbf{d} \in \mathrm{R}^4 | \mathbf{d}^T\mathrm{H}\mathbf{d} \geqq 0\}$ and $\mathrm{S} = \{\mathbf{x} \in \mathcal{K} | \|\mathbf{x}\| = 1\}$. Without loss of generality we may assume $\eta = 0$. There exists a sequence $\{\mathbf{x}_h\}$ in $\mathcal{H}$ and a corresponding sequence $\{\hat{\mathbf{x}}_n\}$ in S, with $\hat{\mathbf{x}}_n = \mathbf{x}_n/\|\mathbf{x}_n\|$, such that

$$\mathbf{x}_n^T\mathrm{M}\mathbf{x}_n = \hat{\mathbf{x}}_n^T\mathrm{M}\hat{\mathbf{x}}_n/\hat{\mathbf{x}}_n^T\mathrm{H}\hat{\mathbf{x}}_n \downarrow \eta \ (,\ n \to \infty) \ . \tag{3.19}$$

Some subsequence $\{\hat{\mathbf{x}}_{n_k}\}$ of $\{\hat{\mathbf{x}}_n\}$ is convergent; let be $\mathbf{d}$ its limit; $\mathbf{d} \in \partial\mathcal{K}$ for otherwise the infimum η would be assumed on $\mathcal{H}$. So $\mathbf{d}^T\mathrm{H}\mathbf{d} = 0$. Hence also $\mathbf{d}^T\mathbf{Md} = 0$. It is clear that $\mathbf{x}^T\mathbf{Mx} \geqq 0$ for each $\mathbf{x} \in \partial\mathcal{K}$. So $h|\partial K$ assumed its minimum in $\mathbf{d}$. Application of Lagranges multiplier method for that minimum gives: there exists a $\lambda \in \mathbf{R}$ such that

$$\mathbf{Md} = \lambda\mathbf{Hd} \ . \tag{3.20}$$

Let be $\mathbf{x} = \mathbf{d} + \mathbf{h} \in \mathrm{S}$. Since $\mathbf{d}^T\mathbf{Md} = \mathbf{d}^T\mathrm{H}\mathbf{d}$ we get with (3.20)

$$\frac{\mathbf{x}^T\mathbf{Mx}}{\mathbf{x}^T\mathrm{H}\mathbf{x}} = \frac{2\lambda(\mathrm{B}\mathbf{d})^T\mathbf{h} + \mathbf{h}^T\mathbf{Mh}}{2(\mathrm{B}\mathbf{d})^T\mathbf{h} + \mathbf{h}^T\mathrm{H}\mathbf{h}} \ . \tag{3.21}$$

Now $(\mathrm{B}\mathbf{d})^T\mathbf{h} > 0$ when $\mathbf{h} \in \mathrm{int}(\mathcal{K})$. Hence

$$\mathbf{x}^T\mathbf{Mx}/\mathbf{x}^T\mathrm{H}\mathbf{x} \to \lambda \quad (,\ \mathbf{x} \in \mathrm{S},\ \mathbf{x} \to \mathbf{d}) \ .$$

So $\lambda = \eta$. □

COROLLARY. If $\mathbf{x}^T\mathbf{Mx} > \eta$ for each $\mathrm{x} \in \mathcal{H}$ then the intersection of $\mathcal{H}$ and the subspace $\{\tau_1(1,1,0,0)^T + \tau_2\mathbf{d} | \tau_1,\tau_2 > 0\}$ is a curve

$$\Gamma : \mathbf{x}(t) = t(d_1+d_2)^{-1}\mathbf{d} + t^{-1}(d_1+d_2)^{-1}(d_2,d_1,-d_3,-d_4)^T$$

along which $\mathbf{x}(t)^T\mathbf{Mx}(t)$ tends to η for $t \to \infty$. □

EXAMPLE 3.1.

Let be

$$A = \begin{bmatrix} 0 & -i & 1 & 2 \\ -i & 2 & 2 & 1+4i \\ 4+i & 1-3i & 5+i & 2+i \\ 1-3i & 10+3i & 2+i & 3+5i \end{bmatrix}$$

Then

$$M = \begin{bmatrix} 20 & 0 & 0 & 40 \\ 0 & 20 & 0 & 40 \\ 0 & 0 & 40 & 0 \\ 40 & 40 & 0 & 120 \end{bmatrix}, \quad H^{-1}M = 40\begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & -3 \end{bmatrix}.$$

$\lambda = -40$ is the fourfold eigenvalue of the pair (M,H). The eigenspace is spanned by the three vectors $(-1,1,0,0)^{\top}$, $(-2,0,0,1)^{\top}$ and $(0,0,1,0)^{\top}$. None linear combination of these vectors is in $\mathcal{H}$; $\mathbf{d} = (\sqrt{3})^{-1}(1,1,0,-1)^{\top}$ satisfies the equality $d_1d_2 - d_3^2 - d_4^2 = 0$. The line $\mathbf{x}(\tau) = \tau(1,1,0,-1)^{\top}$ is the asymptote of the curve $\Gamma : \mathbf{x}(\tau) = (2\tau)^{-1}(\tau^2+1,\ \tau^2+1,\ 0,\ \tau^2-1)^{\top}$ in $\mathcal{H}$. Γ is the intersection of the plane $\{\tau_1(1,1,0,0)^{\top} + \tau_2\mathbf{d} \mid \tau_1,\tau_2 > 0\}$ and $\mathcal{H}$. Along Γ holds $h(\mathbf{x}(\tau)) \to \eta = -40$. Remark that $-40 + \sum\limits_{\ell,m=1}^{2} (|\alpha_{\ell,m}|^2 + |\beta_{\ell,m}|^2) = 168$, being the sum of the squares of the moduli of the eigenvalues of the four non-diagonalizable matrices $A_{\ell,m}$ of A. □

4. PRECONDITIONED PARALLEL NORMREDUCTION

This section describes a preconditioning transformation by unitary matrices, and the following suboptimal normreduction. This combination brings about convergence to normality. In the first step that twofold action results in

$$A' = W^{-1}U^*AUW = W^{-1}(U^*AU)W \tag{4.1}$$

where

$$U = \mathrm{diag}(U_1,\dots,U_k),\ W = \mathrm{diag}(S,\dots,S) \tag{4.2}$$

with

$$U_\ell = \begin{bmatrix} \cos\varphi_\ell & -e^{-i\vartheta_\ell}\sin\varphi_\ell \\ e^{i\vartheta_\ell}\sin\varphi_\ell & \cos\varphi_\ell \end{bmatrix}, \quad \ell = 1,\dots,k \tag{4.3}$$

and

$$S = \begin{bmatrix} p & 0 \\ 0 & s \end{bmatrix}, \quad ps = 1$$

a 2×2 submatrix in the diagonal of shears W.
As in Sameh's algorithm [5] the *rectifier* U is chosen such that grad $g(0,0,0;W^{-1}AW)$ (see (2.11) and (3.17)) has maximal length. For complex A we use theorem 3.4.

THEOREM 4.1. Let be $U = \mathrm{drag}(U_1,\dots,U_k)$, with U_ℓ, $\ell = 1,\dots,k$, as given in (4.3) and let be

$$v_\ell(A) = v_\ell = \begin{bmatrix} c_{2\ell-1,2\ell-1} - c_{2\ell,2\ell} \\ 2c_{2\ell-1,2\ell} \end{bmatrix}, \quad \ell = 1,\dots,k \tag{4.4}$$

where $(c_{i,j}) = C(A) = A^*A - AA^*$. Then the maximum length of grad $g(0,0,0;U^*AU)$ with respect to U equals $\sum_{\ell=1}^{k} |\mathbf{v}_\ell|$ and is assumed for

$$\begin{bmatrix}\cos 2\varphi_\ell \\ \sin 2\varphi_\ell\end{bmatrix} = \begin{cases}(1,0)^T & \text{, if } \mathbf{v}_\ell = \underline{0}, \\ (c_{2\ell-1,2\ell-1} - c_{2\ell,2\ell},\ 2(c_{2\ell-1,2\ell}))|\mathbf{v}_\ell|^{-1}, & \text{if } \mathbf{v}_\ell \neq 0\end{cases} \tag{4.5}$$

with $|\varphi_\ell| \leqq \frac{\pi}{2}$ and

$$e^{i\vartheta} = \begin{cases}1 & \text{, if } c_{2\ell-1,2\ell} = 0 \\ (c_{2\ell,2\ell-1}/|c_{2\ell-1,2\ell}|, & \text{if } c_{2\ell-1,2\ell} \neq 0 .\end{cases} \tag{4.6}$$

Then $g_w(0,0,0;U^*AU) \geqq 0$, $g_u(0,0,0;U^*AU) = g_v(0,0,0;U^*AU) = 0$.

PROOF. It follows from the corollary of theorem 3.4 that grad $g(0,0,0;U^*AU) = \sum_{\ell=1}^{k} (c'_{2\ell-1,2\ell-1} - c'_{2\ell,2\ell},\ 2\mathrm{Re}(c_{2\ell-1,2\ell}),\ 2\mathrm{Im}(c_{2\ell-1,2\ell}))$ where $(c'_{i,j}) = C(U^*AU)$. Since $C(U^*AU) = U^*C(A)U$ we find

$$\begin{bmatrix}c'_{2\ell-1,2\ell-1} - c'_{2\ell,2\ell} \\ e^{i\vartheta}c'_{2\ell-1,2\ell}\end{bmatrix} = \begin{bmatrix}\cos 2\varphi & \sin 2\varphi & \sin 2\varphi \\ -\frac{1}{2}\sin 2\varphi & \cos^2\varphi & -\sin^2\varphi\end{bmatrix}\begin{bmatrix}c_{2\ell-1,2\ell-1} - c_{2\ell,2\ell} \\ e^{i\vartheta}\ c_{2\ell-1,2\ell} \\ e^{-i\vartheta}\ \bar{c}_{2\ell-1,2\ell}\end{bmatrix}$$

With $\cos\varphi$, $\sin\varphi$ and $e^{i\vartheta}$ as given in (4.6) and (4.5)

$$\begin{bmatrix}c'_{2\ell-1,2\ell-1} - c'_{2\ell,2\ell} \\ 2c_{2\ell-1,2\ell}\end{bmatrix} = \mathbf{v}_\ell(U^*AU) = \begin{bmatrix}|\mathbf{v}_\ell(A)| \\ 0\end{bmatrix} . \tag{4.7}$$

With these transformations $U_1,\dots,U_k$ the k vectors $\mathbf{v}_\ell(U^*AU)$ have the same direction, the vectors $\mathbf{v}_\ell(A)$ are *rectified*. Thus

$$|g(0,0,0;U^*AU)| = |\sum_{\ell=1}^{k} \mathbf{v}_\ell(U^*AU)| = \sum_{\ell=1}^{k} |\mathbf{v}_\ell(U^*AU)| = \sum_{\ell=1}^{k} \mathbf{v}_\ell(A)|.$$

Now $g_u(0,0,0;U^*AU) = g_v(0,0,0;U^*AU) = 0$ and $g_w(0,0,0;U^*AU) \geqq 0$. □

THEOREM 4.2. Let be U a unitary matrix as defined in (4.2), (4.3), (4.5) and (4.6). Then there exists a diagonal matrix W such that

$$\|A\|_F^2 - \|W^{-1}U^*AUW\|_F^2 \geqq \frac{1}{8}\|A\|_F^2 \sum_{\ell=1}^{k} \|v_\ell\|^2 \tag{4.8}$$

where v_ℓ as defined in (4.4).

PROOF. Let be $A' = U^*AU$ and W a diagonal of identical diagonal shears $\begin{bmatrix} p & 0 \\ 0 & s \end{bmatrix}$. According to (3.12)

$$\|W^{-1}A'W\|_F^2 = \sum_{\ell,m=1}^{k} (|\alpha'_{\ell,m}|^2 + |\beta'_{\ell,m}|^2 + |\mu'_{\ell,m}|^2 x^{-2} + |\sigma'_{\ell,m}|^2 x^2) ,$$

for the Euclidean parameter z of a diagonal shear equals zero. Let be $\sum^{k}|\sigma'_{\ell,m}|^2 = c_1$, $\sum_{\ell,m}^{k} |\mu'_{\ell,m}|^2 = c_2$.

1. Let be $c_1c_2 \neq 0$. Then $c_1x^2 + c_2x^{-2}$ is minimal for $x = (c_2/c_1)^{\frac{1}{4}}$. With the Euclidean parameters $((c_2,c_1)^{\frac{1}{4}},(c_1/c_2)^{\frac{1}{4}},0)$ the decrease of the Euclidean norm equals

$$\|A\|_F^2 - \|W^{-1}U^*AUW\|_F^2 = (\sqrt{c_1}-\sqrt{c_2})^2 .$$

Since $c_1 - c_2 = \frac{1}{2}\, g_w(0,0,0;A')$

$$\sqrt{c_1} - \sqrt{c_2} = (\sqrt{c_1}+\sqrt{c_2})^{-1}(c_1-c_2) = \tfrac{1}{2}(\sqrt{c_1}+\sqrt{c_2})^{-1} g_w(0,0,0;A^1) .$$

Now $\sqrt{c_1} + \sqrt{c_2} \leqq \sqrt{2}(c_1+c_2)^{\frac{1}{2}} \leqq \sqrt{2}\|A'\|_F = \sqrt{2}\|A\|_F$ and

$$g_w(0,0,0;A') = \sum_{\ell=1}^{k} (c'_{2\ell-1,2\ell-1} - c'_{2\ell,2\ell}) = \sum_{\ell=1}^{k} \|v_\ell(U^*AU)\| \geqq \sum_{\ell=1}^{k} \|v_\ell(A)\|$$

as can be seen from (4.7). Consequently

$$\|A\|_f^2 - \|W^{-1}U^*AUW\|_F^2 \geqq \frac{1}{8}\|A\|_F^{-2}\Big(\sum_{\ell=1}^{k}\|v_\ell(A)\|\Big)^2 \geqq \frac{1}{8}\|A\|_F^{-2}\sum_{\ell=1}^{k}\|v_\ell(A)\|^2.$$

2. In case $c_1 \neq 0$ and $c_2 = 0$, choose x so small that $x^2 \leq 1 - \|A\|_F^{-2}c_1/8$. Then

$$\|A\|_F^2 - \|W^{-1}U^*AUW\|_F^2 = c_1(1-x^2) \geqq \|A\|_F^{-2}c_1^2/8 \geqq \frac{1}{8}\|A\|_F^{-2}\sum_{\ell=1}^{k}\|v_\ell(A)\|^2$$

for $c_1 = \sum_{\ell=1}^{k}(c'_{2\ell-1,2\ell-1} - c'_{2\ell,2\ell}) = \sum_{\ell=1}^{k}\|v_\ell(U^*AU)\| = \sum_{\ell=1}^{k}\|v_\ell(A)\|$.

3. In case $c_2 = 0$ and $c_1 \neq 0$ choose $x^2 \geq (1 - \|A\|_F^{-2}c_2/8)^{-1}$. □

The pivot strategy guaranteeing that the Euclidean norm decreases in sufficient degree for convergence to normality [2,3] will be derived from lower bound (4.8). Therefore we need

THEOREM 4.3. There exists a set of k distinct index pairs (ℓ_j, m_j), with $\ell_j < m_j$, $j = 1,\dots,k$, such that

$$\sum_{j=1}^{k}(c_{\ell_j,\ell_j} - c_{m_j,m_j}) + 4|c_{\ell_j,m_j}|^2) \geqq \frac{4}{n-1}\|C(A)\|_F^2 . \tag{4.9}$$

PROOF. We have $\sum_{\ell\neq m}(c_{\ell,\ell} - c_{m,m})^2 = 2(n-1)\sum_{\ell=1}^{n}c_{\ell,\ell}^2 - 2\sum_{\ell\neq m}^{m}c_{\ell,\ell}c_{m,m}$. But since $\sum_{\ell=1}^{n}c_{\ell,\ell} = 0$, $(\sum_{\ell=1}^{n}c_{\ell,\ell})^2 = \sum_{\ell=1}^{n}c_{\ell,\ell}^2 + \sum_{\ell\neq m}c_{\ell,\ell}c_{m,m} = 0$. Hence for $n \geqq 2$: $\sum_{\ell\neq m}(c_{\ell,\ell} - c_{m,m})^2 = 2n\sum_{\ell=1}^{n}c_{\ell,\ell}^2 \geqq 4\sum_{\ell=1}^{n}c_{\ell,\ell}^2$. Consequently

$$\sum_{\ell\neq m}((c_{\ell,\ell} - c_{m,m})^2 + 4|c_{\ell,m}|^2) \geqq 4\|C(A)\|_E^2 \tag{4.10}$$

Let be Ω the collection of all sets ω of k distinct index pairs (ℓ_j, m_j). The number of sets w in Ω is $n!/(k!2^k)$, and each pair (ℓ, m), with $\ell \neq m$, occurs in $(n-2)!/((k-1)!2^{k-1})$ sets of Ω. Thus

$$\sum_{\omega\in\Omega}\sum_{(\ell,m)\in\omega}((c_{\ell,\ell}-c_{m,m})^2+4|c_{\ell,m}|^2)=\sum_{\ell\neq m}\sum_{\substack{\omega\in\Omega\\(\ell,m)\in\omega}}((c_{\ell,\ell}-c_{m,m})^2+4|c_{\ell,m}|^2)$$
$$=\frac{(n-2)!}{(k-1)!2^{k-1}}\sum_{\ell\neq m}((c_{\ell,\ell}-c_{m,m})^2+4|c_{\ell,m}|^2)\ .$$

Hence the mean of $\Sigma_{(\ell,m)\in\omega}((c_{\ell,\ell}-c_{m,m})^2+4|c_{\ell,m}|^2)$ overall $\omega\in\Omega$ equals

$$\left[\frac{n!}{k!2^k}\right]^{-1}\frac{(n-2)!}{(k-1)!2^{k-1}}\sum_{\ell\neq m}((c_{\ell,\ell}-c_{m,m})^2+4|c_{\ell,m}|^2=$$
$$(n-1)^{-1}\sum_{\ell\neq m}((c_{\ell,\ell}-c_{m,m})^2+4|c_{\ell,m}|^2)\ .$$

This result, together with (4.10), proves the theorem. □

THEOREM 4.4. Let a sequence $\{A^{(j)}\}$, starting with $A^{(0)}=A$, be constructed by

$$A^{(j+1)}=(P^{(j)}U^{(j)}W^{(j)})^{-1}A^{(j)}P^{(j)}U^{(j)}W^{(j)}\ ,\qquad j=1,2,\ldots \tag{4.11}$$

where in each step k disjunct index pairs $(\ell_{1,j},m_{1,j}),\ldots,(\ell_{k,j},m_{k,j})$ are selected according to rule (4.9). $P^{(j)}$ is a permutation with $P(\ell_{1,j},m_{1,j},\ldots,\ell_{k,j},m_{k,j})=(1,2,\ldots,n-1,n)$. $U^{(j)}$ is a preconditioning unitary block diagonal matrix a described in (4.2), (4.3), (4.5) and (4.6) and $\tilde{W}^{(j)}=\mathrm{diag}(x_j^{\frac12},x_j^{-\frac12},\ldots,x_j^{\frac12},x_j^{-\frac12})$ that reduces the Frobenius norm of $(P^{(j)}U^{(j)})^{-1}A^{(j)}P^{(j)}U^{(j)}$ as described in theorem 4.2. Then $\{A^{(j)}\}$ converges to normality.

PROOF. $\{A_F^{(j)}\}$ decreases monotonically and is bounded below. Therefore $\delta_j:=\|A^{(j)}\|_F^2-\|A^{(j+1)}\|_F^2\downarrow 0,\ (,j\to\infty)$. Since by theorem 4.2 and theorem 4.3

$$\delta_j \geqq \frac{1}{8} \|A\|_F^{-2} \sum_{\ell=1}^{k} \|v_\ell((P^{(j)}U^{(j)})^{-1}A^{(j)}P^{(j)}U^{(j)})\|^2$$

$$\geqq \frac{1}{2(n-1)} \|A\|_F^{-2}\|C((P^{(j)}U^{(j)}A^{(j)}P^{(j)}U^{(j)}\|_F^2$$

$$= \frac{1}{2(n-1)} \|A\|_F^{-2}\|C(A^{(j)})\|_F^2 ,$$

we have $C(A^{(j)}) \to 0$ (, $j \to \infty$). □

THEOREM 4.5. Let $\{A^{(j)}\}$ be constructed recursively by

$$A^{(j+1)} = (P^{(j)}U^{(j)}W^{(j)})^{-1}A^{(j)}P^{(j)}U^{(j)}W^{(j)} , \quad j = 1,2,... \tag{4.12}$$

with $P^{(j)}$ and $U^{(j)}$ as in theorem 4.4 but $W^{(j)}$ an optimal parallel norm-reducing shear as described in section 3. Then $\{A^{(j)}\}$ converges to normality.

PROOF. $\delta_j = \|A^{(j)}\|_F^2 - \|A^{(j+1)}\|_F^2 \downarrow 0$ (, $j \to \infty$) for now $W^{(j)}$ is even optimal. As in the proceding theorem $C(A^{(j)}) \to 0$ (, $j \to \infty$). □

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